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Tilt Classifications in Perfect Fluid Cosmology

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Tilt Classifications in Perfect Fluid Cosmology

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We classify all known perfect fluid cosmological solutions of the Einstein equations according to whether they are tilted or non-tilted. A non-tilted universe will have observers who see a homogeneous, isotropic universe with matter at rest with respect to them. A tilted universe will have observers who see matter moving relative to them. These classifications are useful when considering fluid models of the universe in that the Hubble parameter and expansion are observer dependent and can be different in a tilted versus a non-tilted universe. This gives more insight when fitting these models with observations of our real universe. We make these tilt classifications by establishing whether the 4-velocity of each model's fluid is aligned with the normal of the hyper-surfaces of homogeneity spanned by the Killing vectors for the space-time, which we obtain for each solution. These computations are performed using the Differential Geometry software package being developed at Utah State University. We incorporate the Killing vector fields and the tilt classification into a library of solutions to Einstein's Field Equations as part of the package, providing users with access to the solutions and their physical and geometric properties.

I. INTRODUCTION

There are over a thousand known solutions to Einstein's field equations, representing a wide range of physical and mathematical situations. In an effort to make accessing and using these solutions quick and easy, our research group at Utah State University has been compiling all known solutions into a library for *Maple* as part of the Differential Geometry package also developed at Utah State University [2]. Examples of these computations, and how to access this library using *Maple* 18 can be found in the appendix.

By not only including the solutions themselves, but their intrinsic physical and geometric properties as well, this library will provide a powerful tool for research in General Relativity and Differential Geometry. In a few simple keystrokes, one can have access to virtually every known solution, as well as all of their physical and geometric properties, saving time making or finding those calculations. It also makes it possible to search for and find a solution of interest that has whatever physical or geometric properties one might be interested in.

This in turn, gives a fast and easy solution to what has aptly been named the Equivalence Problem. It is centered around whether, given two metric tensors, there exists a coordinate transformation that makes them equivalent. Einstein's theory is based on his postulate that the laws of physics are invariant under coordinate transformations from one inertial reference frame to another. This means that any given solution's physical and geometric properties should also be invariant under coordinate transformation that go from one inertial frame to another. Thus these properties are what uniquely define a solution, and this library will provide the means to identify a known solution based solely off of them.

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A. Perfect Fluid Cosmology

One of the most commonly used models of the universe are perfect fluid solutions to Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa_0 T_{\mu\nu} \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor, R is the Ricci scalar, Λ is the cosmological constant, $T_{\mu\nu}$ is the energy-momentum tensor, $g_{\mu\nu}$ is the metric tensor for the space-time in question, and $\kappa_0 = \frac{8\pi G}{c^4}$. The metric tensor signature used for all the space-times looked at here is (-,+,+,+).

Physically, these field equations imply that all the information (i.e. energy, momentum, curvature) about a given system is contained within the metric tensor $g_{\mu\nu}$, the solution to equation (1). This means that given a metric tensor, all the physical and geometric properties of that space-time can be extracted from it.

Perfect fluid cosmologies treat the universe as being filled with matter that behaves approximately like a perfect fluid, making the heat flux and viscosity terms negligible in the energy momentum tensor that appears in equation (1). This makes solving the field equations far simpler and as a first order approximation, appears to fit well with observations, so most known cosmological solutions are of this form. The $(2, 0)$ version of the energy momentum tensor then is given by

$$T^{\mu\nu} = \Big(\rho + \frac{p}{c^2}\Big)U^\mu U^\nu + p\eta^{\mu\nu} \tag{2}$$

where ρ is the energy density of the fluid, p is the pressure, and U^μ is the 4-vector velocity of the fluid.

The pressure and energy density for most of these models are related by a constant γ in the gamma law equation of state

$$p = \rho(\gamma - 1) \tag{3}$$

Many of these solutions as special cases, can be reduced down to simpler forms with either stiff matter or dust equations of state $p = \rho$ or $p = 0$ respectively[4].

To find the four-velocity of the fluid, if p and ρ are known for the space-time, plugging equation (2) into (1), gives four equations for each of the four indices. The only unknowns are then the four components of the four-velocity, thus making it straightforward to solve for each component.

B. Tilt Classifications in Cosmology

An intrinsic property of each solution is whether or not a homogeneous observer is co-moving with the matter in the universe. If so, the solution is "aligned", otherwise it is "tilted". We assume a universe foliated into space-like hypersurfaces determined by the model's group orbits. Then there can be found two time-like vector fields that define two congruences. The first is the geometric congruence naturally defined as the vector field $\mathbf{N}$, normal to the group orbits and thus orthogonal to the surfaces of transitivity[3]. This vector field must be geodesic, as well as vorticity and acceleration free.

Each solution has unique families of world lines orthogonal to the hypersurfaces of homogeneity spanned by its Killing vector fields, or in other words, a set of observers who see a homogeneous universe. These spanning Killing vector fields, X, are found from the Killing equation given in local coordinates by

$$\nabla_\mu X_\nu + \nabla_\nu X_\mu = 0$$

or more consisely, as the vanishing Lie derivative of the metric tensor

$$\mathcal{L}_X g_{\mu\nu} = 0 \tag{4}$$

The set of vector fields that satisfy (4) form a Lie algebra and are generators of the isometry group G_n, acting on each space-time. They are tangent to the homogeneous hypersurfaces defined by the level sets of the function λ on the given spacetime. This implies that any function $f(\lambda)$ is invariant under the isometry group and thus satisfy

$$\mathcal{L}_X f(\lambda) = 0 \tag{5}$$

This then gives a prescription for finding the geometric congruence defined by the vector field $\mathbf{N}$ (normal to the hypersurfaces of homogeneity), mentioned above. It is defined simply as the gradient of the function λ.

$$\mathbf{N} = g^{\alpha\gamma}\nabla_\gamma \lambda^\beta \tag{6}$$

The second congruence is the matter congruence given as the time-like four-vector velocity $\mathbf{U}$ of the fluid. Unlike $\mathbf{N}$, $\mathbf{U}$ need not be geodesic and may have both acceleration and vorticity[3].

If $\mathbf{N}$ and $\mathbf{U}$ are linearly dependent, then the solution is aligned. If they are linearly independent then the solution is tilted. A quick and easy way of checking this is by taking the wedge product of the two. An aligned solution will necessarily have a vanishing wedge product.

$$Aligned : \mathbf{U} \wedge \mathbf{N} = 0 \tag{7}$$

A tilted solution will always satisfy.

$$Tilted : \mathbf{U} \wedge \mathbf{N} \neq 0 \tag{8}$$

Tilt tends to complicate the equations represented in (1) due to diagonalization methods and theorems not being applicable to tilted solutions. This being the case, there are far fewer tilted solutions known than aligned cases. However, tilted cases are of interest for multiple cosmological considerations. For example, some tilted solutions permit periods of rapid inflation in cosmological history that their aligned counterparts do not. They are able to represent more dynamic and sophisticated models that could potentially better fit descriptions of our real universe. Tilt can also affect the predicted behavior of the expanding universe and its ultimate fate. Within these models, it is possible for an observer experiencing tilt to see a collapsing universe, while an aligned observer could see the universe infinitely expanding. The in-depth details and derivations of these cosmological considerations are beyond the scope of this paper but are explored by Coley, Hervik and Lim [3].

Another reason tilt classifications are of interest goes back to the Equivalence Problem discussed earlier. Because tilt is intrinsic, it is one of the defining properties that can be used to help identify a particular solution. So if a solution is transformed into a new coordinate system, its tilt classification is invariant.

II. EXAMPLES

The collection of solutions used for this project are taken from chapter 14 of the extensive solution encyclopedia published by Stephani [4]. This includes 65 perfect fluid cosmological solutions, and of those 65 we found that a total of 8 were tilted models. The following sections give examples of the classification process and the Killing vector fields for titled and aligned solutions for each class of metric that was analyzed.

A. Robertson-Walker Cosmologies

These space-times have line elements of the form[4]

$$ds^2 = -dt^2 + a^2(t)[dr^2 + \Sigma^2(r,k)(d\xi^2 + sin^2(\xi)d\phi^2)]$$

All the solutions of this class have been found to be aligned.

As an example, we take the solution given as [4]

$$g = -dt \otimes dt + a(t)^2 dr \otimes dr + a(t)^2 sin(r)^2 (d\theta \otimes d\theta + sin(\theta)^2 d\phi \otimes d\phi) \quad (9)$$

To find the Killing vector fields for this space-time, we start with a general vector field with components that are functions of the coordinates t, r, θ, and ϕ

$$V = T(t, r, \theta, \phi)\partial_t + R(t, r, \theta, \phi)\partial_r + \Theta(t, r, \theta, \phi)\partial_\theta + \Phi(t, r, \theta, \phi)\partial_\phi \quad (10)$$

Taking the Lie derivative of this vector field using the metric tensor in (9) gives a system of partial differential equations for T, R, Θ, and Φ, whose solutions can then be plugged into (10) to give the Killing vector fields

$$\begin{aligned}
X_1 &= -\,cos(\phi)sin(\theta)\partial_r - \frac{cos(\phi)cos(\theta)}{tan(r)}\partial_\theta + \frac{sin(\phi)}{tan(r)sin(\theta)}\partial_\phi \\
X_2 &= sin(\phi)sin(\theta)\partial_r + \frac{sin(\phi)cos(\theta)}{tan(r)}\partial\theta + \frac{cos(\phi)}{tan(r)sin(\theta)}\partial\phi \\
X_3 &= -\,\partial_\phi \\
X_4 &= cos(\phi)\partial_\theta - \frac{sin(\phi)}{tan(\theta)}\partial_\phi \\
X_5 &= -\,sin(\phi)\partial_\theta - \frac{cos(\phi)}{tan(\theta)}\partial_\phi \\
X_6 &= -\,cos(\theta)\partial_r + \frac{sin(\theta)}{tan(r)}\partial_\theta
\end{aligned} \quad (11)$$

It is straightforward to check that each of these satisfy equation (4) and thus generate the isometry group for this space-time. Once we have the Killing vector fields we can find the function, λ, which satisfies equation (5). Again taking a general function $f(t, r, \theta, \phi)$, we apply equation (5) to each Killing vector given in (11) which gives another system of partial differential equations whose solutions in this case are simply any function $f(t)$, dependent only on t. This gives $\lambda = t$ in (5), so using the function

$$f(t) = t$$

in equation (7) gives the Normal vector field as

$$\mathbf{N} = -\partial_t$$

Comparing this with the fluid's four-velocity found here to be

$$\mathbf{U} = \partial_t$$

makes it clear by inspection, that $\mathbf{U}$ and $\mathbf{N}$ are linearly dependent and thus the solution is aligned.

B. Aligned cosmologies on a $\mathbf{G}_4$ on $\mathbf{S}_3$

These solutions' line elements are of the form[4]

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)[dy^2 + \Sigma^2(y, k)dz^2] \quad (12)$$

These solutions obey the gamma-law equation of state given above, and the energy density is given in the form of

$$\rho = \frac{\rho_0}{(AB^2)^\gamma} \quad (13)$$

where ρ_0 is a constant.

Within this class of metrics, both Stephani's published version for Vajk and Eltrgroth's [1] solution (Stephani 14.14)[4], as well as the original paper by Vajk and Eltgroth's [1] contain errors that causes the metric tensor to not satisfy (1). Stephani's error most likely came from a mistake in his simplification process using hyperbolic trigonometric identities that created a wrong exponent in the g_{00} component of the metric. We found the correct metric tensor, which satisfies (1), with energy density given by (13), to be

$$\begin{aligned}
g = &-\frac{16(sinh(u)cosh(u))^{\frac{-2\gamma}{\gamma-2}}}{3M(\gamma-2)^2} du \otimes du \\
&+ cosh(u)^{-\frac{4}{\gamma-2}} sinh(u)^{\frac{4}{3(\gamma-2)}} dx \otimes dx \\
&+ sinh(u)^{-\frac{8}{3(\gamma-2)}} (dy \otimes dy + y^2 dz \otimes dz)
\end{aligned} \quad (14)$$

as a general solution for $1 < \gamma < 2$.

The Killing vector fields for this case then are found to be

$$\begin{aligned}
X_1 &= \partial_z \\
X_2 &= -cos(z)\partial_y + (sin(z)/y)\partial_z \\
X_3 &= sin(z)\partial_y + (cos(z)/y)\partial_z \\
X_4 &= \left(\frac{cosh(u)tanh(u)}{sinh(u)}\right)^{\frac{4}{\gamma-2}} \partial_x
\end{aligned} \quad (15)$$

Using the same method applied in the previous section we find that these Killing vector fields are tangent to the hypersurfaces of $f = u = const$, so again using equation (6) we get

$$\mathbf{N} = -\frac{3(sinh(u)cosh(u))^{\frac{2\gamma}{\gamma-2}} M(\gamma-2)^2}{16}\partial_u$$

The four velocity for this case is

$$\mathbf{U} = \left(\frac{3M(\gamma-2)^2}{-16(sinh(u)cosh(u))^{\frac{-2\gamma}{\gamma-2}}}\right)^{1/2} \partial_u$$

Using these results in equation (7) shows that the wedge product vanishes so the solution is aligned.

The mistake in Vajk and Eltrgroth's [1] original paper made their proposed solution (given in the form of (12) with energy density (13)), not satisfy (1). To make their solution, as presented in their paper, satisfy (1), it was necessary to modify their energy density given in (13) by rescaling it using an integration constant raised to a power of $\gamma - 1$.

C. Tilted cosmologies on a $\mathbf{G}_4$ on $\mathbf{S}_3$

A tilted example of a solution under this class [4] is

$$g = \frac{b^2}{(U(x+t))^2}[-(\dot{U}(x+t))^2 dt \otimes dt/a^2 + dx \otimes dx + e^{-2x}(dy \otimes dy + dz \otimes dz)] \quad (16)$$

Where b is a constant and the dot represents differentiation with respect to $x+t$ such that $\ddot{U}(x+t) + \dot{U}(x+t) + (U(x+t))^2 = 0$. To find the Killing vectors here, a simple coordinate transformation is necessary to reduce computation time. We use the transformation

$$u = x+t, t = t, y = y, z = z$$

Then transforming the metric tensor in (16) gives

$$\tilde{g} = \frac{b^2}{(U(u))^2}[du \otimes du - du \otimes dt - dt \otimes du - (\frac{1}{a^2}\left(\frac{dU}{du}\right)^2 - 1)dt \otimes dt + e^{2t-2u}(dy \otimes dy + dz \otimes dz)]$$

The Killing vectors for this solution are

$$\begin{aligned} X_1 &= \frac{1}{b^2}(-z\partial_y + y\partial_z) \\ X_2 &= \frac{1}{b^2}(-\partial_t + y\partial_y + z\partial_z) \\ X_3 &= \frac{1}{b^2}\partial_z \\ X_4 &= \frac{1}{b^2}\partial_y \end{aligned} \quad (17)$$

Pushing these vector fields back to the original coordinates gives the final Killing vector fields as

$$\begin{aligned} X_1 &= \frac{1}{b^2}(-z\partial_y + y\partial_z) \\ X_2 &= \frac{1}{b^2}(-\partial_t + y\partial_y + z\partial_z + \partial_x) \\ X_3 &= \frac{1}{b^2}\partial_z \\ X_4 &= \frac{1}{b^2}\partial_y \end{aligned} \quad (18)$$

The differential equations these generate from equation (5) give functions on the hypersurface

$$f = x+t$$

whose exterior derivative is then

$$df = dx + dt$$

giving the orthogonal vector field

$$\mathbf{N} = -\frac{a^2(U(x+t))^2}{b^2\dot{U}^2}\partial_t + \frac{(U(x+t))^2}{b^2}\partial_x$$

The fluid's four velocity in this case is

$$\mathbf{U} = \frac{aU(x+t)}{b\dot{U}}\partial_t$$

This gives the wedge product as

$$\mathbf{N} \wedge \mathbf{U} = -\frac{a(U(x+t))^3}{b^3\dot{U}}\partial_t \wedge \partial_x$$

indicating a tilted solution.

D. Aligned cosmologies on a $\mathbf{G}_3$ on $\mathbf{S}_3$

Here we use the example [4]

$$g = -dt \otimes dt + S(t)^2 F(t)^{2cos(\psi)} dx \otimes dx + S(t)^2 F(t)^{-2sin(\psi+\frac{\pi}{6})} dy \otimes dy + S(t)^2 F(t)^{-2cos(\psi+\frac{\pi}{3})} dz \otimes dz \quad (19)$$

The Killing vector fields here are of the particularly simple form

$$\begin{aligned} X_1 &= \partial_x \\ X_2 &= \partial_y \\ X_3 &= \partial_z \end{aligned} \quad (20)$$

It is easy to check that these are tangent to the hypersurfaces of $t = const$, making the normal vector field

$$\mathbf{N} = -\partial_t$$

Comparing this with the fluids four-velocity

$$\mathbf{U} = \partial_t$$

makes it clear that the two are linearly dependent, making the solution aligned.

E. Tilted cosmologies on a $\mathbf{G}_3$ on $\mathbf{S}_3$

An example of a tilted solution of this class is given by [4]

$$g = -e^{-2ax} dt \otimes dt + dx \otimes dx + e^{-2ax}(t^{n+1} dy \otimes dy + t^{n-1} dz \otimes dz) \quad (21)$$

Here the symmetries are represented by the Killing vector fields

$$\begin{aligned} X_1 &= -\frac{1}{a}\partial_x + \frac{1}{2}y(n-1)\partial_y - \frac{1}{2}z(n+1)\partial_z - t\partial_t \\ X_2 &= \partial_y \\ X_3 &= \partial_z \end{aligned} \quad (22)$$

These have orbits represented by $f = te^{-ax}$, giving an exterior derivative of

$$df = -ate^{-ax} dx + e^{-ax} dt$$

. From this we find that the Normal vector field is

$$\mathbf{N} = -ate^{-ax}\partial_x - e^{ax}\partial_t$$

This is linearly independent of the fluid's four velocity

$$\mathbf{U} = e^{ax}\partial_t$$

thus giving a tilted solution.

[1] J. P. Vajk and P. G. Eltgroth, Journal of Mathematical Physics (1970).
[2] C. G. Torre and D. Krongos, Journal of Mathematical Physics **56** (2015).
[3] A. A. Coley, S. Hervik, and W. C. Lim, Classical and Quantum Gravity **23** (2006).
[4] H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers, and E. Herlt, *Exact Solutions to Einstien's Field Equations*, edited by 2nd Edition (Cambridge University Press, 2003).

Appendix: Maple Computational Methods

Here we show a few of the above computational processes using our Differential Geometry package for Maple 18.

Example 1

```
P > restart;
```

First we need to include all the needed packages.

```
> with(DifferentialGeometry): with(Tensor): with(Library): with(GroupActions):
```

For this example we use the Metric given by equation (9) above. This metric is taken from Chapter 14 of Stephani's book (see ref. [4]). In Stephani's book it is labeled equation 14.12 a), so in our library of solutions it can be found under the label Stephani [14,12,1].

Here we retrieve our metric and it's four-velocity.

```
P > g,U:= op(Retrieve("Stephani", 1,[14, 12, 1], manifoldname = P, output = ["Metric",
    "FourVelocity"]));
```

$$g, U := -dt \otimes dt + \_a(t)^2\, dr \otimes dr + \_a(t)^2 \sin(r)^2\, d\theta \otimes d\theta + \_a(t)^2 \sin(r)^2 \sin(\theta)^2\, d\phi \otimes d\phi,\ \partial_t \tag{1.1}$$

Next we use the Killing vectors command to find the Killing vector fields for the solution. This command works well for many of the more basic metric tensors. However, often for more complicated solutions, it causes the computation to time out, and it becomes necessary to use other means to find the Killing vector fields as we will see in later examples.
The simplify command includes the most basic trigonometric identities and will simplify the output using them where possible.

```
P > KV :=simplify(value(KillingVectors(g)));
```

$$\begin{aligned} KV := \Bigg[& \frac{1}{2\sin(r)^3\sin(\theta)^3}\Big(\sqrt{2}\cos(\phi)\sqrt{-2+2\cos(2r)}\,\big(\cos(r)^2\cos(\theta)^4 - 2\cos(r)^2\cos(\theta)^2 - \cos(\theta)^4 \\ & + \cos(r)^2 + 2\cos(\theta)^2 - 1\big)\Big)\,\partial_r \\ & - \frac{\sqrt{2}\cos(r)\cos(\phi)\cos(\theta)\sqrt{-2+2\cos(2r)}\,\big(\cos(r)^2\cos(\theta)^2 - \cos(r)^2 - \cos(\theta)^2 + 1\big)}{2\sin(r)^4\sin(\theta)^2}\,\partial_\theta \\ & + \frac{\sqrt{2}\sqrt{-2+2\cos(2r)}\cos(r)\sin(\phi)}{2\sin(r)^2\sin(\theta)}\,\partial_\phi, \\ & - \frac{1}{2\sin(r)^3\sin(\theta)^3}\Big(\sqrt{2}\sin(\phi)\sqrt{-2+2\cos(2r)}\,\big(\cos(r)^2\cos(\theta)^4 - 2\cos(r)^2\cos(\theta)^2 - \cos(\theta)^4 \\ & + \cos(r)^2 + 2\cos(\theta)^2 - 1\big)\Big)\,\partial_r \\ & + \frac{\sqrt{2}\cos(r)\sin(\phi)\cos(\theta)\sqrt{-2+2\cos(2r)}\,\big(\cos(r)^2\cos(\theta)^2 - \cos(r)^2 - \cos(\theta)^2 + 1\big)}{2\sin(r)^4\sin(\theta)^2}\,\partial_\theta \\ & + \frac{\sqrt{2}\sqrt{-2+2\cos(2r)}\cos(r)\cos(\phi)}{2\sin(r)^2\sin(\theta)}\,\partial_\phi,\ \frac{\sqrt{-1+\cos(2r)}\sqrt{-1+\cos(2\theta)}}{\sin(r)\sin(\theta)}\,\partial_\phi,\ 0\,\partial_r \\ & + \frac{\cos(\phi)\sqrt{-2+2\cos(2r)}\,\big(\cos(r)^2\cos(\theta)^2 - \cos(r)^2 - \cos(\theta)^2 + 1\big)}{\sin(r)^3\sin(\theta)\sqrt{-\sin(\theta)^2}}\,\partial_\theta \\ & + \frac{\sqrt{2}\sqrt{-2+2\cos(2r)}\sqrt{-1+\cos(2\theta)}\sin(\phi)\cos(\theta)}{2\sin(r)\sin(\theta)^2}\,\partial_\phi,\ 0\,\partial_r \end{aligned} \tag{1.2}$$

$$- \frac{\sin(\phi)\sqrt{-2+2\cos(2r)}\left(\cos(r)^2\cos(\theta)^2-\cos(r)^2-\cos(\theta)^2+1\right)}{\sin(r)^3\sin(\theta)\sqrt{-\sin(\theta)^2}}\,\partial_\theta$$
$$+ \frac{\sqrt{2}\sqrt{-2+2\cos(2r)}\sqrt{-1+\cos(2\theta)}\cos(\phi)\cos(\theta)}{2\sin(r)\sin(\theta)^2}\,\partial_\phi,$$
$$\frac{1}{2\sin(r)^4\sin(\theta)^2}\left(\cos(\theta)\left(\cos(r)^2\cos(2r)\cos(\theta)^2-\cos(r)^2\cos(\theta)^2-\cos(r)^2\cos(2r)\right.\right.$$
$$\left.\left.-\cos(2r)\cos(\theta)^2+\cos(r)^2+\cos(\theta)^2+\cos(2r)-1\right)\right)\partial_r$$
$$-\frac{1}{2\sin(r)^5\sin(\theta)}\left(\cos(r)\left(\cos(r)^2\cos(2r)\cos(\theta)^2-\cos(r)^2\cos(\theta)^2-\cos(r)^2\cos(2r)\right.\right.$$
$$\left.\left.-\cos(2r)\cos(\theta)^2+\cos(r)^2+\cos(\theta)^2+\cos(2r)-1\right)\right)\partial_\theta\Big]$$

By inspection, it is clear that the above output can be simplified further using trigonometric identities that maple doesn't recognize. In the following steps we manually define these identities and use them to further condense the Killing Vector fields.

First we define the identities.

```
P > trigID:=[cos(theta)^2=1-sin(theta)^2,cos(r)^2=1-sin(r)^2,sqrt(2*cos(theta)^2-2)=sqrt(-2*
    sin(theta)^2),sqrt(-1+cos(2*theta))=sqrt(-2*sin(theta)^2),sqrt(-2+2*cos(2*r))=sqrt(-4*
    sin(r)^2),sqrt(cos(r)^2-1)=I*sin(r),-1+cos(2*r)=-2*sin(r)^2];
```

$$trigID := \Big[\cos(\theta)^2 = 1-\sin(\theta)^2,\ \cos(r)^2 = 1-\sin(r)^2,\ \sqrt{-2+2\cos(\theta)^2} = \sqrt{-2\sin(\theta)^2},\ \sqrt{-1+\cos(2\theta)} = \sqrt{-2\sin(\theta)^2},\ \sqrt{-2+2\cos(2r)} = 2\sqrt{-\sin(r)^2},\ \sqrt{\cos(r)^2-1} = \mathrm{I}\sin(r),\ -1+\cos(2r) = -2\sin(r)^2\Big] \quad \textbf{(1.3)}$$

Next we use the map command to evaluate KV using these defined identities. Often it becomes necessary to do this multiple times, using one identity at a time. The symbolic simplify command is also often useful. This steps requires some trial and error to find the best order in which to make the most complete simplifications. The evalDG command needs to be included to ensure that Maple knows to continue to treat these equations as contravariant vector fields.

```
P > KV1:=evalDG(map(eval,simplify(map(eval,simplify(map(eval,KV,trigID)),trigID[6]),
    symbolic),trigID[7]));
```

$$KV1 := \Big[-\mathrm{I}\sqrt{2}\cos(\phi)\sin(\theta)\,\partial_r - \frac{\mathrm{I}\sqrt{2}\cos(r)\cos(\phi)\cos(\theta)}{\sin(r)}\,\partial_\theta + \frac{\mathrm{I}\sqrt{2}\cos(r)\sin(\phi)}{\sin(r)\sin(\theta)}\,\partial_\phi,$$
$$\mathrm{I}\sqrt{2}\sin(\phi)\sin(\theta)\,\partial_r + \frac{\mathrm{I}\sqrt{2}\cos(r)\sin(\phi)\cos(\theta)}{\sin(r)}\,\partial_\theta + \frac{\mathrm{I}\sqrt{2}\cos(r)\cos(\phi)}{\sin(r)\sin(\theta)}\,\partial_\phi,\ -2\,\partial_\phi,\ 0\,\partial_r + 2\cos(\phi)\,\partial_\theta$$
$$-\frac{2\sin(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi,\ 0\,\partial_r - 2\sin(\phi)\,\partial_\theta - \frac{2\cos(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi,\ -\cos(\theta)\,\partial_r + \frac{\cos(r)\sin(\theta)}{\sin(r)}\,\partial_\theta\Big] \quad \textbf{(1.4)}$$

Now we have more manageable vector fields, but they can still be reduced further. Because constants do not effect derivatives, any that appear in every term of a Killing vector can be divided out. Because Maple has trouble factoring vector fields, it is easiest to get rid of the constants in the first two vector fields in KV1 simply by forcing maple to redefine the imaginary number I using the map command. This is done below.

```
P > KV2:=evalDG(([map(eval,KV1[1],I=1/sqrt(2)),map(eval,KV1[2],I=1/sqrt(2)),KV1[3]/(2),KV1
    [4]/2,KV1[5]/2,KV1[6]]));
```

$$KV2 := \Big[-\sin(\theta)\cos(\phi)\,\partial_r - \frac{\cos(r)\cos(\phi)\cos(\theta)}{\sin(r)}\,\partial_\theta + \frac{\cos(r)\sin(\phi)}{\sin(r)\sin(\theta)}\,\partial_\phi,\ \sin(\phi)\sin(\theta)\,\partial_r$$
$$+\frac{\cos(r)\sin(\phi)\cos(\theta)}{\sin(r)}\,\partial_\theta + \frac{\cos(r)\cos(\phi)}{\sin(r)\sin(\theta)}\,\partial_\phi,\ -\partial_\phi,\ \cos(\phi)\,\partial_\theta - \frac{\sin(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi,\ -\sin(\phi)\,\partial_\theta$$
$$-\frac{\cos(\phi)\cos(\theta)}{\sin(\theta)}\,\partial_\phi,\ -\cos(\theta)\,\partial_r + \frac{\cos(r)\sin(\theta)}{\sin(r)}\,\partial_\theta\Big] \quad \textbf{(1.5)}$$

Now that we have condensed versions of the Killing vector fields, we double check that they are correct by using them to take the Lie Derivative of the metric tensor to ensure that it vanishes.

```
P > LD:=simplify(LieDerivative(KV1,g),symbolic);
```

$$LD := [0\,dt\otimes dt,\ 0\,dt\otimes dt,\ 0\,dt\otimes dt,\ 0\,dt\otimes dt,\ 0\,dt\otimes dt,\ 0\,dt\otimes dt] \quad \textbf{(1.6)}$$

Now that we have our Killing Vector fields we need to find the orbits so that we can define the normal vector field. This can be done automatically using the following command:

```
P > InvariantGeometricObjectFields(KV2,[1]);
```

$$\_F1(t) \tag{1.7}$$

This can also be found manually, however the above command uses the same process to calculate the result, so doing it manually is usually unnecessary. We have included it here for information purposes.

We define a general function of the coordinates defining the manifold.

```
P > f:=F(t,r,theta,phi);
```

$$f := F(t, r, \theta, \phi) \tag{1.8}$$

We then take the Lie derivative of it using the Killing vector fields.

```
P > LD:=LieDerivative(KV2,f);
```

$$LD := \left[-\sin(\theta)\cos(\phi)\left(\frac{\partial}{\partial r}F(t,r,\theta,\phi)\right) - \frac{\cos(r)\cos(\phi)\cos(\theta)\left(\frac{\partial}{\partial \theta}F(t,r,\theta,\phi)\right)}{\sin(r)} + \frac{\cos(r)\sin(\phi)\left(\frac{\partial}{\partial \phi}F(t,r,\theta,\phi)\right)}{\sin(r)\sin(\theta)}, \sin(\phi)\sin(\theta)\left(\frac{\partial}{\partial r}F(t,r,\theta,\phi)\right) + \frac{\cos(r)\sin(\phi)\cos(\theta)\left(\frac{\partial}{\partial \theta}F(t,r,\theta,\phi)\right)}{\sin(r)} + \frac{\cos(r)\cos(\phi)\left(\frac{\partial}{\partial \phi}F(t,r,\theta,\phi)\right)}{\sin(r)\sin(\theta)}, -\left(\frac{\partial}{\partial \phi}F(t,r,\theta,\phi)\right), \cos(\phi)\left(\frac{\partial}{\partial \theta}F(t,r,\theta,\phi)\right) - \frac{\sin(\phi)\cos(\theta)\left(\frac{\partial}{\partial \phi}F(t,r,\theta,\phi)\right)}{\sin(\theta)}, -\sin(\phi)\left(\frac{\partial}{\partial \theta}F(t,r,\theta,\phi)\right) - \frac{\cos(\phi)\cos(\theta)\left(\frac{\partial}{\partial \phi}F(t,r,\theta,\phi)\right)}{\sin(\theta)}, -\cos(\theta)\left(\frac{\partial}{\partial r}F(t,r,\theta,\phi)\right) + \frac{\cos(r)\sin(\theta)\left(\frac{\partial}{\partial \theta}F(t,r,\theta,\phi)\right)}{\sin(r)} \right] \tag{1.9}$$

Then solving this system of partial differential equations gives

```
P > pdsolve(LD);
```

$$\{F(t, r, \theta, \phi) = \_F1(t)\} \tag{1.10}$$

This is the same result as the InvariantGeometricObjectFields command gave.

Now we double check that the function f=t is in fact invariant.

```
P > LieDerivative(KV2, t);
```

$$[0, 0, 0, 0, 0, 0] \tag{1.11}$$

We are finally ready to find the normal vector field. We first want the exterior derivative of f=t.

```
P > df:=ExteriorDerivative(t);
```

$$df := dt \tag{1.12}$$

We then take this one-form and raise its index using the inverse metric tensor, giving us the normal vector field.

```
P > N := RaiseLowerIndices(InverseMetric(g), dt, [1]);
```

$$N := -\partial_t \tag{1.13}$$

Taking the wedge product of N and U gives

```
P > evalDG(N &w U);
```

$$0\,\partial_t \wedge \partial_r \tag{1.14}$$

indicating an aligned Solution.

Example II

In the next two examples we give solutions whose Killing vectors Maple is unable to compute with the KillingVectors command.

The first is the solution (given above in equation (14)) found under the label Stephani [14,24,1].

```
> restart;
P > g,U := op(Retrieve("Stephani", 1,[14 ,24 ,1], manifoldname = P, output = ["Metric",
"FourVelocity"]));
```

$$g, U := -\frac{\_c^2\,D(\_U)(t+x)^2}{\_U(t+x)^2\,\_a^2}\,dt \otimes dt + \frac{\_c^2}{\_U(t+x)^2}\,dx \otimes dx + \frac{\_c^2\,e^{-2x}}{\_U(t+x)^2}\,dy \otimes dy + \frac{\_c^2\,e^{-2x}}{\_U(t+x)^2}\,dz \otimes dz, \tag{2.1}$$

$$\frac{\_U(t+x)\,\_a}{\_c\,\mathrm{D}(\_U)(t+x)}\,\partial_t$$

For this case, as discussed above, a coordinate transformation is the easiest way to find the Killing vector fields. So first we define the new manifold.

```
P > DGsetup([u,t,y,z],M);
```

$$\textit{Manifold: } M \tag{2.2}$$

Then we define the transformation.

```
M > T:=Transformation(P,M,[u=t+x,t=t,y=y,z=z]);
```

$$T := u = t + x, t = t, y = y, z = z \tag{2.3}$$

```
P > T1:=InverseTransformation(T);
```

$$T1 := t = t, x = u - t, y = y, z = z \tag{2.4}$$

We transform the metric tensor into the new coordinate system.

```
M > g1:=PushPullTensor(T,T1,g);
```

$$g1 := \frac{\_c^2}{\_U(u)^2}\,du \otimes du - \frac{\_c^2}{\_U(u)^2}\,du \otimes dt - \frac{\_c^2}{\_U(u)^2}\,dt \otimes du - \frac{\_c^2\left(\mathrm{D}(\_U)(u)^2 - \_a^2\right)}{\_U(u)^2\,\_a^2}\,dt \otimes dt + \frac{\_c^2\,\mathrm{e}^{-2u+2t}}{\_U(u)^2}\,dy \otimes dy + \frac{\_c^2\,\mathrm{e}^{-2u+2t}}{\_U(u)^2}\,dz \otimes dz \tag{2.5}$$

With the metric tensor now expressed in new coordinates, the KillingVector command is able to find that solution's Killing vector fields expressed in the transformed coordinates.

```
M > KV1:=KillingVectors(g1);
```

$$KV1 := \left[-\frac{z}{\_c^2}\,\partial_y + \frac{y}{\_c^2}\,\partial_z, -\frac{1}{\_c^2}\,\partial_t + \frac{y}{\_c^2}\,\partial_y + \frac{z}{\_c^2}\,\partial_z, \frac{1}{\_c^2}\,\partial_z, \frac{1}{\_c^2}\,\partial_y\right] \tag{2.6}$$

It is straightforward to push the above Killing vectors back into the original coordinates and check that they are correct.

```
M > KV:=PushPullTensor(T1,T,KV1);
```

$$KV := \left[-\frac{z}{\_c^2}\,\partial_y + \frac{y}{\_c^2}\,\partial_z, -\frac{1}{\_c^2}\,\partial_t + \frac{1}{\_c^2}\,\partial_x + \frac{y}{\_c^2}\,\partial_y + \frac{z}{\_c^2}\,\partial_z, \frac{1}{\_c^2}\,\partial_z, \frac{1}{\_c^2}\,\partial_y\right] \tag{2.7}$$

```
P > LieDerivative(KV,g);
```

$$[0\,dt \otimes dt, 0\,dt \otimes dt, 0\,dt \otimes dt, 0\,dt \otimes dt] \tag{2.8}$$

We use the InvariantGeometricObject command and proceed to calculate N in the same fashion used in the previous example.

```
P > InvariantGeometricObjectFields(KV,[1]);
```

$$\_F1(t+x) \tag{2.9}$$

```
P > df:=ExteriorDerivative(t+x);
```

$$df := dt + dx \tag{2.10}$$

```
P > N := RaiseLowerIndices(InverseMetric(g), df, [1]);
```

$$N := -\frac{\_U(t+x)^2\,\_a^2}{\_c^2\,\mathrm{D}(\_U)(t+x)^2}\,\partial_t + \frac{\_U(t+x)^2}{\_c^2}\,\partial_x \tag{2.11}$$

We take the wedge product of N and U.

```
P > evalDG(N &w U);
```

$$-\frac{\_U(t+x)^3\,\_a}{\_c^3\,\mathrm{D}(\_U)(t+x)}\,\partial_t \wedge \partial_x \tag{2.12}$$

Thus the solution is tilted.

Example III

As the last example, we look at a solution whose Killing vector fields must be found manually using a general vector field. We use the solution Stephani [14,42,1].

```
> restart;
P > g,U := op(Retrieve("Stephani", 1,[14 ,42,1], manifoldname = P, output = ["Metric",
"FourVelocity"]));
```

$$g, U := -dt \otimes dt + \left(\_k^2\,\_m^2\,t^2 + \_k^2\,t^2\right) dx \otimes dx + \_m\,\_k\,t^{1-\_q+\_s}\,\mathrm{e}^x\,dx \otimes dy + \_m\,\_k\,t^{1-\_q+\_s}\,\mathrm{e}^x\,dy \otimes dx + t^{-2\_q+2\_s}\,\mathrm{e}^{2x}\,dy \otimes dy + t^{2\_q+2\_s}\,\mathrm{e}^{-2x}\,dz \otimes dz, \_u4\,\partial_t + \frac{\_u1}{\_k\,t}\,\partial_x - \mathrm{e}^{-x}\,t^{\_q-\_s}\left(\_m\,\_u1 - \_u2\right)\partial_y \tag{3.1}$$

If we try and compute the Killing vector fields using the KillingVector command, we find that the computation times out. With no immediately obvious coordinate transformations visible, we find the Killing vectors manually. We first find out how many Killing vector fields to expect. The following command takes the metric tensor, and the point (1,0,0,0) defined on the Manifold P, and returns the structure equations for that space-time determined by commutations of the Killing vectors fields.

```
P > IsometryAlgebraData(g,[t=1,x=0,y=0,z=0]);
```

$$[e1, e2] = -e2,\ [e1, e3] = e3,\ [e2, e3] = 0 \tag{3.2}$$

While the vector fields themselves are left undefined, it is immediately apparent how many should be expected. The above result shows three possible vector fields, e1, e2, and e3.

Next we define a general vector field whose components are functions of the four coordinates.

```
P > V:=evalDG(X(t,x,y,z)*D_x+Y(t,x,y,z)*D_y+Z(t,x,y,z)*D_z+T(t,x,y,z)*D_t);
```

$$V := T(t, x, y, z)\, \partial_t + X(t, x, y, z)\, \partial_x + Y(t, x, y, z)\, \partial_y + Z(t, x, y, z)\, \partial_z \tag{3.3}$$

We take the Lie derivative of V, then extract the resulting coefficients, giving us a set of partial differential equations.

```
P > LD:=LieDerivative(V,g);
```

$$\begin{aligned}
LD := & -2 \left(\frac{\partial}{\partial t} T(t,x,y,z)\right) dt \otimes dt + \Big(\Big(\frac{\partial}{\partial t} X(t,x,y,z)\Big) _k^2\, _m^2\, t^2 + \Big(\frac{\partial}{\partial t} Y(t,x,y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} \\
& + \Big(\frac{\partial}{\partial t} X(t,x,y,z)\Big) _k^2\, t^2 - \Big(\frac{\partial}{\partial x} T(t,x,y,z)\Big)\Big) dt \otimes dx + \Big(-\Big(\frac{\partial}{\partial y} T(t,x,y,z)\Big) + \Big(\frac{\partial}{\partial t} X(t,x,y, \\
& z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} + \Big(\frac{\partial}{\partial t} Y(t,x,y,z)\Big) t^{-2_q+2_s}\, \mathrm{e}^{2x}\Big) dt \otimes dy + \Big(-\Big(\frac{\partial}{\partial z} T(t,x,y,z)\Big) + \Big(\frac{\partial}{\partial t} Z(t,x, \\
& y,z)\Big) t^{2_q+2_s}\, \mathrm{e}^{-2x}\Big) dt \otimes dz + \Big(\Big(\frac{\partial}{\partial t} X(t,x,y,z)\Big) _k^2\, _m^2\, t^2 + \Big(\frac{\partial}{\partial t} Y(t,x,y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} \\
& + \Big(\frac{\partial}{\partial t} X(t,x,y,z)\Big) _k^2\, t^2 - \Big(\frac{\partial}{\partial x} T(t,x,y,z)\Big)\Big) dx \otimes dt + 2\, _k \Big(\Big(\frac{\partial}{\partial x} X(t,x,y,z)\Big) _k\, _m^2\, t^2 + T(t,x,y, \\
& z)\, _k\, _m^2\, t + \Big(\frac{\partial}{\partial x} X(t,x,y,z)\Big) _k\, t^2 + \mathrm{e}^{x}\, t^{1-_q+_s} \Big(\frac{\partial}{\partial x} Y(t,x,y,z)\Big) _m + T(t,x,y,z)\, _k\, t\Big) dx \otimes dx + \\
& \Big(\Big(\frac{\partial}{\partial y} X(t,x,y,z)\Big) _k^2\, _m^2\, t^2 - T(t,x,y,z)\, _m\, _k\, t^{-_q+_s}\, \mathrm{e}^{x}\, _q + T(t,x,y,z)\, _m\, _k\, t^{-_q+_s}\, \mathrm{e}^{x}\, _s + T(t,x, \\
& y,z)\, _m\, _k\, t^{-_q+_s}\, \mathrm{e}^{x} + \Big(\frac{\partial}{\partial x} X(t,x,y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} + \Big(\frac{\partial}{\partial y} X(t,x,y,z)\Big) _k^2\, t^2 + X(t,x,y, \\
& z)\, _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} + \Big(\frac{\partial}{\partial x} Y(t,x,y,z)\Big) t^{-2_q+2_s}\, \mathrm{e}^{2x} + \Big(\frac{\partial}{\partial y} Y(t,x,y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x}\Big) dx \otimes dy \\
& + \Big(\Big(\Big(\frac{\partial}{\partial z} X(t,x,y,z)\Big) \mathrm{e}^{2x}\, _k^2\, _m^2\, t^2 + \Big(\frac{\partial}{\partial z} X(t,x,y,z)\Big) \mathrm{e}^{2x}\, _k^2\, t^2 + \mathrm{e}^{3x}\, t^{1-_q+_s} \Big(\frac{\partial}{\partial z} Y(t,x,y, \\
& z)\Big) _k\, _m + \Big(\frac{\partial}{\partial x} Z(t,x,y,z)\Big) t^{2_q+2_s}\Big) \mathrm{e}^{-2x}\, dx \otimes dz + \Big(-\Big(\frac{\partial}{\partial y} T(t,x,y,z)\Big) + \Big(\frac{\partial}{\partial t} X(t,x,y, \\
& z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} + \Big(\frac{\partial}{\partial t} Y(t,x,y,z)\Big) t^{-2_q+2_s}\, \mathrm{e}^{2x}\Big) dy \otimes dt + \Big(\Big(\frac{\partial}{\partial y} X(t,x,y,z)\Big) _k^2\, _m^2\, t^2 - T(t, \\
& x,y,z)\, _m\, _k\, t^{-_q+_s}\, \mathrm{e}^{x}\, _q + T(t,x,y,z)\, _m\, _k\, t^{-_q+_s}\, \mathrm{e}^{x}\, _s + T(t,x,y,z)\, _m\, _k\, t^{-_q+_s}\, \mathrm{e}^{x} + \Big(\frac{\partial}{\partial x} X(t,x, \\
& y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} + \Big(\frac{\partial}{\partial y} X(t,x,y,z)\Big) _k^2\, t^2 + X(t,x,y,z)\, _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} + \Big(\frac{\partial}{\partial x} Y(t,x,y, \\
& z)\Big) t^{-2_q+2_s}\, \mathrm{e}^{2x} + \Big(\frac{\partial}{\partial y} Y(t,x,y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x}\Big) dy \otimes dx + \Big(-2\, T(t,x,y,z)\, t^{-2_q+2_s-1}\, \mathrm{e}^{2x}\, _q \\
& + 2\, T(t,x,y,z)\, t^{-2_q+2_s-1}\, \mathrm{e}^{2x}\, _s + 2 \Big(\frac{\partial}{\partial y} X(t,x,y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{x} + 2\, X(t,x,y, \\
& z)\, t^{-2_q+2_s}\, \mathrm{e}^{2x} + 2 \Big(\frac{\partial}{\partial y} Y(t,x,y,z)\Big) t^{-2_q+2_s}\, \mathrm{e}^{2x}\Big) dy \otimes dy + \Big(\Big(\Big(\frac{\partial}{\partial z} X(t,x,y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{3x} \\
& + \Big(\frac{\partial}{\partial z} Y(t,x,y,z)\Big) t^{-2_q+2_s}\, \mathrm{e}^{4x} + \Big(\frac{\partial}{\partial y} Z(t,x,y,z)\Big) t^{2_q+2_s}\Big) \mathrm{e}^{-2x}\, dy \otimes dz + \Big(-\Big(\frac{\partial}{\partial z} T(t,x,y,z)\Big) \\
& + \Big(\frac{\partial}{\partial t} Z(t,x,y,z)\Big) t^{2_q+2_s}\, \mathrm{e}^{-2x}\Big) dz \otimes dt + \Big(\Big(\Big(\frac{\partial}{\partial z} X(t,x,y,z)\Big) \mathrm{e}^{2x}\, _k^2\, _m^2\, t^2 + \Big(\frac{\partial}{\partial z} X(t,x,y, \\
& z)\Big) \mathrm{e}^{2x}\, _k^2\, t^2 + \mathrm{e}^{3x}\, t^{1-_q+_s} \Big(\frac{\partial}{\partial z} Y(t,x,y,z)\Big) _k\, _m + \Big(\frac{\partial}{\partial x} Z(t,x,y,z)\Big) t^{2_q+2_s}\Big) \mathrm{e}^{-2x}\, dz \otimes dx \\
& + \Big(\Big(\Big(\frac{\partial}{\partial z} X(t,x,y,z)\Big) _m\, _k\, t^{1-_q+_s}\, \mathrm{e}^{3x} + \Big(\frac{\partial}{\partial z} Y(t,x,y,z)\Big) t^{-2_q+2_s}\, \mathrm{e}^{4x} + \Big(\frac{\partial}{\partial y} Z(t,x,y, \\
& z)\Big) t^{2_q+2_s}\Big) \mathrm{e}^{-2x}\, dz \otimes dy + 2\, \mathrm{e}^{-2x} \Big(T(t,x,y,z)\, t^{2_q+2_s-1}\, _q + T(t,x,y,z)\, t^{2_q+2_s-1}\, _s + \Big(\frac{\partial}{\partial z}
\end{aligned} \tag{3.4}$$

$$Z(t,x,y,z)\Big)\,t^{2\_q+2\_s} - X(t,x,y,z)\,t^{2\_q+2\_s}\Big)\,dz \otimes dz$$

```
P > LDC:=DGinfo(LD,"CoefficientSet");
```

$$LDC := \Big\{\Big(\Big(\frac{\partial}{\partial z}X(t,x,y,z)\Big)\_m\,\_k\,t^{1-\_q+\_s}\,e^{3x} + \Big(\frac{\partial}{\partial z}Y(t,x,y,z)\Big)t^{-2\_q+2\_s}\,e^{4x} + \Big(\frac{\partial}{\partial y}Z(t,x,y,z)\Big)t^{2\_q+2\_s}\Big)e^{-2x}, \Big(\Big(\frac{\partial}{\partial z}X(t,x,y,z)\Big)e^{2x}\,\_k^2\,\_m^2\,t^2 + \Big(\frac{\partial}{\partial z}X(t,x,y,z)\Big)e^{2x}\,\_k^2\,t^2 + e^{3x}\,t^{1-\_q+\_s}\Big(\frac{\partial}{\partial z}Y(t,x,y,z)\Big)\_k\,\_m + \Big(\frac{\partial}{\partial x}Z(t,x,y,z)\Big)t^{2\_q+2\_s}\Big)e^{-2x}, 2\,\_k\Big(\Big(\frac{\partial}{\partial x}X(t,x,y,z)\Big)\_k\,\_m^2\,t^2 + T(t,x,y,z)\,\_k\,\_m^2\,t + \Big(\frac{\partial}{\partial x}X(t,x,y,z)\Big)\_k\,t^2 + e^{x}\,t^{1-\_q+\_s}\Big(\frac{\partial}{\partial x}Y(t,x,y,z)\Big)\_m + T(t,x,y,z)\,\_k\,t\Big), 2\,e^{-2x}\Big(T(t,x,y,z)\,t^{2\_q+2\_s-1}\,\_q + T(t,x,y,z)\,t^{2\_q+2\_s-1}\,\_s + \Big(\frac{\partial}{\partial z}Z(t,x,y,z)\Big)t^{2\_q+2\_s} - X(t,x,y,z)\,t^{2\_q+2\_s}\Big), -2\Big(\frac{\partial}{\partial t}T(t,x,y,z)\Big), -\Big(\frac{\partial}{\partial z}T(t,x,y,z)\Big) + \Big(\frac{\partial}{\partial t}Z(t,x,y,z)\Big)t^{2\_q+2\_s}\,e^{-2x}, -\Big(\frac{\partial}{\partial y}T(t,x,y,z)\Big) + \Big(\frac{\partial}{\partial t}X(t,x,y,z)\Big)\_m\,\_k\,t^{1-\_q+\_s}\,e^{x} + \Big(\frac{\partial}{\partial t}Y(t,x,y,z)\Big)t^{-2\_q+2\_s}\,e^{2x}, \Big(\frac{\partial}{\partial t}X(t,x,y,z)\Big)\_k^2\,\_m^2\,t^2 + \Big(\frac{\partial}{\partial t}Y(t,x,y,z)\Big)\_m\,\_k\,t^{1-\_q+\_s}\,e^{x} + \Big(\frac{\partial}{\partial t}X(t,x,y,z)\Big)\_k^2\,t^2 - \Big(\frac{\partial}{\partial x}T(t,x,y,z)\Big), -2\,T(t,x,y,z)\,t^{-2\_q+2\_s-1}\,e^{2x}\,\_q + 2\,T(t,x,y,z)\,t^{-2\_q+2\_s-1}\,e^{2x}\,\_s + 2\Big(\frac{\partial}{\partial y}X(t,x,y,z)\Big)\_m\,\_k\,t^{1-\_q+\_s}\,e^{x} + 2\,X(t,x,y,z)\,t^{-2\_q+2\_s}\,e^{2x} + 2\Big(\frac{\partial}{\partial y}Y(t,x,y,z)\Big)t^{-2\_q+2\_s}\,e^{2x}, \Big(\frac{\partial}{\partial y}X(t,x,y,z)\Big)\_k^2\,\_m^2\,t^2 - T(t,x,y,z)\,\_m\,\_k\,t^{-\_q+\_s}\,e^{x}\,\_q + T(t,x,y,z)\,\_m\,\_k\,t^{-\_q+\_s}\,e^{x}\,\_s + T(t,x,y,z)\,\_m\,\_k\,t^{-\_q+\_s}\,e^{x} + \Big(\frac{\partial}{\partial x}X(t,x,y,z)\Big)\_m\,\_k\,t^{1-\_q+\_s}\,e^{x} + \Big(\frac{\partial}{\partial y}X(t,x,y,z)\Big)\_k^2\,t^2 + X(t,x,y,z)\,\_m\,\_k\,t^{1-\_q+\_s}\,e^{x} + \Big(\frac{\partial}{\partial x}Y(t,x,y,z)\Big)t^{-2\_q+2\_s}\,e^{2x} + \Big(\frac{\partial}{\partial y}Y(t,x,y,z)\Big)\_m\,\_k\,t^{1-\_q+\_s}\,e^{x}\Big\} \qquad \textbf{(3.5)}$$

We then use the PDETools package to solve these equations, giving the components of V.

```
P > Vcomp:=PDETools:-Solve(LDC,[T(t,x,y,z),X(t,x,y,z),Y(t,x,y,z),Z(t,x,y,z)]);
```

$$Vcomp := \{T(t,x,y,z) = 0, X(t,x,y,z) = \_C1, Y(t,x,y,z) = -\_C1\,y + \_C3, Z(t,x,y,z) = \_C1\,z + \_C2\}, \Big\{T(t,x,y,z) = \Big((\_m^2\,\_q - \_m^2\,\_s + \_m^2 + \_q - \_s - 1)(\_q - \_s)\,\_C4\,e^{-\frac{x(-1+\_q-\_s)}{\_m^2\,\_q^2 - 2\,\_m^2\,\_q\,\_s + \_m^2\,\_s^2 + \_m^2\,\_q - \_m^2\,\_s + \_q^2 - 2\,\_q\,\_s + \_s^2 + 1}}\Big)\Big/\Big(\_k\,\_m\,(\_m^2\,\_q^2 - 2\,\_m^2\,\_q\,\_s + \_m^2\,\_s^2 + \_m^2\,\_q - \_m^2\,\_s + \_q^2 - 2\,\_q\,\_s + \_s^2 + 1)\Big), X(t,x,y,z) = \frac{1}{t\,\_m\,\_k}\Big(\_C1\,\_m\,\_k\,t + \_C4\,e^{-\frac{x(-1+\_q-\_s)}{\_m^2\,\_q^2 - 2\,\_m^2\,\_q\,\_s + \_m^2\,\_s^2 + \_m^2\,\_q - \_m^2\,\_s + \_q^2 - 2\,\_q\,\_s + \_s^2 + 1}}\,\_q - \_C4\,e^{-\frac{x(-1+\_q-\_s)}{\_m^2\,\_q^2 - 2\,\_m^2\,\_q\,\_s + \_m^2\,\_s^2 + \_m^2\,\_q - \_m^2\,\_s + \_q^2 - 2\,\_q\,\_s + \_s^2 + 1}}\,\_s\Big), Y(t,x,y,z) = -\_C1\,y + \_C3 + \_C4\,t^{\_q-\_s}\,e^{-\frac{(1+\_q-\_s)(\_q-\_s)(\_m^2+1)\,x}{\_m^2\,\_q^2 - 2\,\_m^2\,\_q\,\_s + \_m^2\,\_s^2 + \_m^2\,\_q - \_m^2\,\_s + \_q^2 - 2\,\_q\,\_s + \_s^2 + 1}}, Z(t,x,y,z) = \_C1\,z + \_C2\Big\}, \Big\{T(t,x,y,z) = \frac{2\,\_C4\,(\_q - \_s)\,e^{x}}{\_m\,\_k\,(-1 + \_q - \_s)}, X(t,x,y,z) = \qquad \textbf{(3.6)}$$

$$-\frac{-\_C1\ \_m\_k\_q\,t+\_C1\ \_m\_k\_s\,t+\_C1\ \_m\_k\,t+2\ \_C4\,e^{x}\,\_q-2\ \_C4\,e^{x}\,\_s}{\_m\_k\,(-1+\_q-\_s)\,t},\ Y(t,x,y,z)=-\_C1\,y+\_C3$$
$$+\_C4\,t^{-\_q-\_s},\ Z(t,x,y,z)=\_C1\,z+\_C2\Big\},\ \Big\{T(t,x,y,z)=-\frac{2\ \_C4\,e^{-x}}{(-1+\_q-\_s)\,\_k\_m},\ X(t,x,y,z)=$$
$$-\frac{-\_C1\ \_m\_k\_q\,t+\_C1\ \_m\_k\_s\,t+\_C1\ \_m\_k\,t+2\,e^{-x}\,\_C4\,\_q-2\,e^{-x}\,\_C4\,\_s}{\_m\_k\,(-1+\_q-\_s)\,t},\ Y(t,x,y,z)=-\_C1\,y$$
$$+\_C3+\_C4\,t^{-\_q-\_s}\,e^{-2x},\ Z(t,x,y,z)=\_C1\,z+\_C2\Big\},\ \Big\{T(t,x,y,z)=\frac{\_C4}{\_m\_k},\ X(t,x,y,z)$$
$$=\frac{\_C1\ \_k\_m\,t+\_C4\,\_q-\_C4\,\_s}{t\,\_m\_k},\ Y(t,x,y,z)=-\_C1\,y+\_C3+\_C4\,t^{-\_q-\_s}\,e^{-x},\ Z(t,x,y,z)=\_C1\,z$$
$$+\_C2\Big\}$$

The results give us multiple sets of solutions with the first looking the most promising.

We plug the first set of solutions into V.

```
P > KVtot:=eval(V,Vcomp[1]);
```

$$KVtot := 0\,\partial_t+\_C1\,\partial_x-(\_C1\,y-\_C3)\,\partial_y+(\_C1\,z+\_C2)\,\partial_z \tag{3.7}$$

KVtot is a linear combination of the Killing vector fields, so we can find each individual vector field by setting one constant equal to 1 in KVtot, and the rest zero.

```
P > KV1:=eval(KVtot,[_C1=1,_C2=0,_C3=0,_C4=0]);
```

$$KV1 := 0\,\partial_t+\partial_x-y\,\partial_y+z\,\partial_z \tag{3.8}$$

```
P > KV2:=eval(KVtot,[_C1=0,_C2=1,_C3=0,_C4=0]);
```

$$KV2 := 0\,\partial_t+0\,\partial_x+0\,\partial_y+\partial_z \tag{3.9}$$

```
P > KV3:=eval(KVtot,[_C1=0,_C2=0,_C3=1,_C4=0]);
```

$$KV3 := 0\,\partial_t+0\,\partial_x+\partial_y+0\,\partial_z \tag{3.10}$$

```
P > KV4:=eval(KVtot,[_C1=0,_C2=0,_C3=0,_C4=1]);
```

$$KV4 := 0\,\partial_t+0\,\partial_x+0\,\partial_y+0\,\partial_z \tag{3.11}$$

This results in three non-null vector fields, as expected from our isometry algebra data. We can check that we indeed have the correct Killing vector fields.

```
P > KV:=[KV1,KV2,KV3];
```

$$KV := [0\,\partial_t+\partial_x-y\,\partial_y+z\,\partial_z,\ 0\,\partial_t+0\,\partial_x+0\,\partial_y+\partial_z,\ 0\,\partial_t+0\,\partial_x+\partial_y+0\,\partial_z] \tag{3.12}$$

```
P > simplify((LieDerivative(KV,g)));
```

$$[0\,dt\otimes dt,\ 0\,dt\otimes dt,\ 0\,dt\otimes dt] \tag{3.13}$$

```
P > InvariantGeometricObjectFields(KV,[1]);
```

$$\_F1(t) \tag{3.14}$$

```
P > LieDerivative(KV,f(t));
```

$$[0,0,0] \tag{3.15}$$

```
P > N := RaiseLowerIndices(InverseMetric(g),dt,[1]);
```

$$N := -\partial_t \tag{3.16}$$

```
P > evalDG(N &w U);
```

$$-\frac{\_u1}{\_k\,t}\,\partial_t\wedge\partial_x+e^{-x}\,t^{-\_q-\_s}\,(\_m\,\_u1-\_u2)\,\partial_t\wedge\partial_y \tag{3.17}$$

So we have a tilted solution.

Accessing the tilt classifications and Killing vector fields directly

The results of this project have been programmed into the library of solutions included with the differential geometry package. Thus every cosmological solutions' Killing vector fields and tilt classification can be accessed directly without the need of any computations, as demonstrated below.

```
P > restart;
P > op(Retrieve("Stephani", 1,[14,42,1], manifoldname = P, output = ["KillingVectors",
    "TertiaryDescription"]));
```

$$[\partial_x-y\,\partial_y+z\,\partial_z,\ \partial_z,\ \partial_y],\ ["\text{Tilted}"] \tag{4.1}$$